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Paper No. : Classical Mechanics

Module : d'Alembert's Principle and Lagrange's Equations

Development Team

Principal Investigator	Prof. Vinay Gupta, Department of Physics and Astrophysics, University of Delhi, Delhi
Paper Coordinator	Ashok Goyal, Department of Physics and Astrophysics, University of Delhi, Delhi
Content Writer	Ashok Goyal, Department of Physics and Astrophysics, University of Delhi, Delhi
Content Reviewer	

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Subject Name	Physics
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d‘Alembert’s Principle and Lagrange’s Equations

Contents:

1. Learning Objectives :

- Learn about a formalism in mechanics based on d’Alembert’s Principle.
- You will learn that this formalism is particularly convenient for constrained systems.

1. Introduction

Newtonian mechanics is based on Newton's laws of motion. In many practical problems it becomes difficult to set up Newton's equation and solve them particularly in the presence of constraints. In such cases an alternative formalism in needed to simplify the analysis of the system. In this unit we would discuss one such alternative formalism based on D'Alembert's Principle. In this context we will first introduce the concept of Virtual work.

2. Principle of virtual work

Let a system of N particle is specified by the coordinates $\{\boldsymbol{r_i}\}$. Virtual displacement is defined as an imaginary infinitesimal displacement $\{\delta\boldsymbol{r_i}\}$ at a fixed given time t while the displacement satisfies the constraints and the generalized velocities $\{\dot{q}_k\}$ are kept fixed at time t. The virtual displacements when represented in generalized coordinates automatically satisfy the constraints. The relation between the infinitesimal displacement of the generalized coordinates and the virtual displacement is given by

$$\delta\boldsymbol{r_i} = \sum_{j} \frac{\partial \boldsymbol{r_i}}{\partial q_{\boldsymbol{j}}} \delta q_j \qquad (2.1)$$

Let $\boldsymbol{F_i}$ be the external applied force and the force of constraints on the ith particle. As the system is in equilibrium the virtual work which is defined as the work done as the system by all the forces that act on the system as the system undergoes virtual displacement $\{\delta\boldsymbol{r_i}\}$ is

$$\delta W = \sum_i \boldsymbol{F_i} \cdot \delta\boldsymbol{r_i} \qquad (2.2)$$

where

$$\boldsymbol{F_i} = \boldsymbol{F_i^{\mathbf{app}}} + \boldsymbol{F_i^{\mathbf{const}}}$$

We now restrict ourselves to systems for which the net virtual work done by the constraint forces is zero i.e.

$$\sum_i \boldsymbol{F_i^{\mathbf{const}}} \cdot \delta\boldsymbol{r_i} = 0 \qquad (2.3)$$

Therefore the virtual work done on the system is given by

$$\delta W = \sum_i \boldsymbol{F_i^{\mathbf{app}}} \cdot \delta\boldsymbol{r_i} \qquad (2.4)$$

Thus the total virtual work done on the system is only by the applied forces. This is the 'principle of virtual work'.

3. d'Alembert's Principle

The total force acting on a particle is the sum of applied and constraint forces. The virtual work done by the constraint forces is zero and therefore, as we saw, the virtual work done on the system is

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$$\delta W = \sum_i \boldsymbol{F_i} \cdot \delta \boldsymbol{r_i}$$

Where we have dropped the superscript 'applied' from the force. From Newton's second law of motion rate of change of momentum of a particle is equal to the applied force. Thus

$$\boldsymbol{F_i} = \frac{d\boldsymbol{p_i}}{dt} = \dot{\boldsymbol{p}}_{\boldsymbol{i}} \tag{2.5}$$

We then have

$$\delta W = \sum \dot{\boldsymbol{p}}_{\boldsymbol{i}} \cdot \delta \boldsymbol{r_i} \tag{2.6}$$

and

$$\sum_i (\boldsymbol{F_i} - \dot{\boldsymbol{p}}_{\boldsymbol{i}}) \cdot \delta \boldsymbol{r_i} = 0 \tag{2.7}$$

This is the d'Alembert's Principle. It says that the rate of change of momentum is determined only by the non-constraint forces. The principle can be generalized to relate the generalized forces to the rate of change of momenta. In terms of generalized coordinates, we have

$$\delta W = \sum_j T_j \, \delta q_j = \sum_i \dot{\boldsymbol{p}}_{\boldsymbol{i}} \cdot \delta \boldsymbol{r_i} = \sum_{ij} \dot{\boldsymbol{p}}_{\boldsymbol{i}} \frac{\partial \boldsymbol{r_i}}{\partial q_j} \delta q_i \tag{2.8}$$

where the generalized for

$$\Im_j = \sum_{ij} \boldsymbol{F_i} \cdot \frac{\partial \boldsymbol{r_i}}{\partial q_j} = \frac{\delta W}{\delta q_j} = \sum_{ij} \dot{\boldsymbol{p}}_{\boldsymbol{i}} \cdot \frac{\partial \boldsymbol{r_i}}{\partial q_j} \tag{2.9}$$

In equation (2.8) and (2.9) above in runs over from 1 to N and j from 1 to 3N-k.

Thus the generalized force on the jth particle can be calculated from the non-constraint applied force alone.

4. Lagrange's Equation of Motion

The kinetic energy T of a dynamical system is defined as

$$T \equiv \frac{1}{2} \sum_i m_i \, \dot{\boldsymbol{r}}_{\boldsymbol{i}} \cdot \dot{\boldsymbol{r}}_{\boldsymbol{i}} = T\left(\delta q_j \{\dot{q}_j\}; t\right) \tag{2.10}$$

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$$T \equiv \frac{1}{2}\sum_i m_i\, \dot{\boldsymbol{r}}_{\boldsymbol{i}} \cdot \dot{\boldsymbol{r}}_{\boldsymbol{i}} = T\big(\delta q_j \{\dot{q}_j\}; t\big) \tag{2.10}$$

This functional relationship of T can be seen by remembering that the position coordinate $\boldsymbol{r}_{\boldsymbol{i}}$ can be written in terms of generalized coordinates and $\dot{\boldsymbol{r}}_{\boldsymbol{i}}$ is obtained by the total derivative of $\dot{\boldsymbol{r}}_{\boldsymbol{i}}$ with time and is a function of $\dot{q}_j$. We can thus write the partial derivative of T as

$$\frac{\partial T}{\partial q_j} = \sum_i m_i \frac{\partial \dot{\boldsymbol{r}}_{\boldsymbol{i}}}{\partial q_j} = \sum_i \dot{\boldsymbol{p}}_{\boldsymbol{i}} \cdot \frac{\partial \dot{\boldsymbol{r}}_{\boldsymbol{i}}}{\partial q_j} \tag{2.11}$$

$$\frac{\partial T}{\partial \dot{q}_j} = \sum_i m_i \dot{\boldsymbol{r}}_{\boldsymbol{i}} \cdot \frac{\partial \dot{\boldsymbol{r}}_{\boldsymbol{i}}}{\partial \dot{q}_j} = \sum_i \dot{\boldsymbol{p}}_{\boldsymbol{i}} \cdot \frac{\partial \dot{\boldsymbol{r}}_{\boldsymbol{i}}}{\partial q_j} \tag{2.12}$$

This happens because

$$\frac{\partial \dot{\boldsymbol{r}}_{\boldsymbol{i}}}{\partial q_j} = \frac{\partial \dot{\boldsymbol{r}}_{\boldsymbol{i}}}{\partial q_0} \tag{2.13}$$

Which can be easily proved by voting that

$$\delta \dot{\boldsymbol{r}}_{\boldsymbol{i}} = \sum_j \dot{\boldsymbol{p}}_{\boldsymbol{i}} \cdot \frac{\partial \dot{\boldsymbol{r}}_{\boldsymbol{i}}}{\partial q_j} \delta q_j \tag{2.14}$$

Now

$$\dot{\boldsymbol{r}}_{\boldsymbol{i}} = \sum_i \frac{\partial \dot{\boldsymbol{r}}_{\boldsymbol{i}}}{\partial \dot{q}_j} \frac{dq_i}{dt} + \frac{\partial \boldsymbol{r}_{\boldsymbol{i}}}{\partial t} \tag{2.15}$$

Taking the partial derivative w.r.t. $\dot{q}_k$

$$\frac{\partial \dot{\boldsymbol{r}}_{\boldsymbol{i}}}{\partial \dot{q}_k} = \sum_i \frac{\partial \boldsymbol{r}_{\boldsymbol{i}}}{\partial q_j} \delta_{jk} = \frac{\partial \dot{\boldsymbol{r}}_{\boldsymbol{i}}}{\partial q_k} \tag{2.16}$$

Differentiating equ. (2.12) w.r.t., we get

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) = \sum_i \dot{\boldsymbol{p}}_{\boldsymbol{i}} \cdot \frac{\partial \boldsymbol{r}_{\boldsymbol{i}}}{\partial q_j} + \sum_i \boldsymbol{p}_{\boldsymbol{i}} \cdot \frac{d}{dt}\left(\frac{\partial \boldsymbol{r}_{\boldsymbol{i}}}{\partial q_j}\right) \tag{2.17}$$

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The first term can be identified as the generalized forces F_j acting on the jth particle. In the second term, we have

$$\frac{d}{dt}\left(\frac{\partial \boldsymbol{r}_i}{\partial q_j}\right) = \sum_k \frac{\partial}{\partial q_k}\left(\frac{\partial \boldsymbol{r}_i}{\partial q_j}\right)\dot{q}_k + \frac{d}{dt}\left(\frac{\partial \boldsymbol{r}_i}{\partial q_j}\right)$$

$$= \frac{\partial}{\partial q_j}\left(\sum_k \frac{\partial \boldsymbol{r}_i}{\partial q_k}\dot{q}_k + \frac{\partial \boldsymbol{r}_i}{\partial t}\right) = \frac{\partial \dot{\boldsymbol{r}}_i}{\partial q_j} \tag{2.18}$$

$$\therefore \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) = \frac{\partial T}{\partial q_j} + F_j$$

We thus have the generalized equation of motion

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} = F_j \tag{2.19}$$

If we now specialize to conservative forces that is the forces that can be obtained from a potential function U.

$$F_v = -\boldsymbol{\nabla}_i U(\{\boldsymbol{r}_k\}) \tag{2.20}$$

The generalized force

$$F_j = \sum_i \boldsymbol{F}_i \cdot \frac{\partial \boldsymbol{r}_i}{\partial q_j} = \sum_i -\boldsymbol{\nabla}_i U(\{\boldsymbol{r}_k\}) \cdot \frac{\partial \boldsymbol{r}_i}{\partial q_j}$$

$$F_j = -\frac{\partial U(\{q_k\}, t)}{\partial q_j} \tag{2.21}$$

The generalized equation of motion obtained by using d'Alembert's Principle expressed in generalized coordinates is given by

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) - \frac{\partial T}{\partial q_k} = -\frac{\partial U}{\partial q_k} \tag{2.22}$$

Define the Lagrangian

$$L = T - U \tag{2.23}$$

For holonomic constraints we have

$$\frac{\partial U}{\partial \dot{q}_k} = 0$$

This allows us to write the above equation in terms of the Lagrangian as

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = 0 \tag{2.24}$$

This is the Euler – Lagrange Equation for each generalized coordinates q_k .

5. Time and velocity dependent Potentials:

In defining the Lagrangian we confined ourselves to potential energy which is time independent. We may have situations where the potential energy even though depends on time but is conservative at each instant of time a for example in the case of planetary motion about the centre of mass. If the potential is velocity dependent the Euler-Lagrange equations may not hold in general. In case we have generalized force that can be written in terms of velocity dependent potentials called generalized potential energy $U(\{q_k\},\{\dot{q}_j\})$ through

$$F_j = -\frac{\partial U}{\partial q_j} + \frac{d}{dt}\left(\frac{\partial U}{\partial \dot{q}_j}\right) \tag{2.25}$$

The Euler – Lagrange equations still hold for $L = T - U$. U defined in this way is not the potential energy in the conventional sense of being the work done by a generalized force. That, it cannot be calculated from the line integral of the generalized force but still allows us to use the concept of Lagrangian to obtain the Euler-Lagrange equations of motion. As an example of velocity dependent force, we will take the example of Lorentz force on a charged particle and obtain the corresponding Lagrange equation.

Lorentz-force Equation:

The equation of motion of a charged particle of mass m and charge q in the presence of electro – magnetic field is given by

$$m\frac{d^2\boldsymbol{r}}{dt^2} = q\left[\boldsymbol{E} + \tfrac{1}{C}\boldsymbol{V} \times \boldsymbol{B}\right] \tag{2.26}$$

where $\boldsymbol{V}$ is the velocity of the particle. The electric and magnetic fields can be expressed in terms of scalar potential and the vector potential **A** through the well known relations from the study of electricity and magnetism.

$$\boldsymbol{E} = -\boldsymbol{\nabla}\phi - \frac{1}{C}\frac{\partial \boldsymbol{A}}{\partial t} \tag{2.27}$$

and

$$\boldsymbol{B} = Curl\boldsymbol{A} \tag{2.28}$$

In terms of X component of the force, equ. (2.26) can be written as

$$m\ddot{x} = q\left[E_x + \tfrac{1}{C}(\boldsymbol{V} \times \boldsymbol{B})_{\boldsymbol{x}}\right]$$

$$= q\left[-\frac{\partial\phi}{\partial x} - \frac{1}{C}\frac{\partial A_x}{\partial t} + \frac{1}{C}\big(\boldsymbol{V} \times (\boldsymbol{\nabla} \times \boldsymbol{B})\big)_{\boldsymbol{x}}\right] \tag{2.29}$$

The x-component of $-\boldsymbol{V} \times (\boldsymbol{\nabla} \times \boldsymbol{B})$ is given by

$$\big(\boldsymbol{V} \times (\boldsymbol{\nabla} \times \boldsymbol{A})\big)_{\boldsymbol{x}} = v_{\boldsymbol{y}}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) - \boldsymbol{v_z}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)$$

Thus

$$m\ddot{x} = q\left[-\frac{\partial\phi}{\partial x} - \frac{1}{C}\frac{\partial A_x}{\partial t} + \frac{1}{C}\left\{v_{\boldsymbol{y}}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) - \boldsymbol{v_z}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\right\}\right] \tag{2.30}$$

Since vector potential depend on the coordinates and time i.e.

$$\boldsymbol{A} = \boldsymbol{A}(x, y, z, t)$$

We have

$$\frac{1}{C}\frac{dA_x}{dt} = \frac{1}{C}\frac{\partial A_x}{\partial t} + \frac{1}{C}\left(v_{\boldsymbol{x}}\frac{\partial A_x}{\partial x} + v_{\boldsymbol{y}}\frac{\partial A_y}{\partial y} + v_z\frac{\partial A_x}{\partial z}\right) \tag{2.31}$$

And substracting $v_{\boldsymbol{x}}\frac{\partial A_x}{\partial x}$ substituting for $\frac{\partial A_x}{\partial t}$ in (2.29) from (2.30), and adding we get

$$m\ddot{x} = q\left[-\frac{\partial\phi}{\partial x} - \frac{1}{C}\frac{\partial A_x}{\partial t} + \frac{1}{C}\left(v_{\boldsymbol{x}}\frac{\partial A_x}{\partial x} + v_{\boldsymbol{y}}\frac{\partial A_y}{\partial y} + v_z\frac{\partial A_x}{\partial z}\right)\right]$$

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$$= q\left[-\frac{\partial\phi}{\partial x} - \frac{1}{C}\frac{\partial A_x}{\partial t} + \frac{1}{C}\boldsymbol{V}\cdot\frac{\partial \boldsymbol{A}}{\partial x}\right]$$

$$= q\left[-\frac{\partial}{\partial x}\left(\phi - \frac{1}{C}\boldsymbol{V}\cdot\boldsymbol{A}\right) + \frac{d}{dt}\frac{\partial}{\partial v_x}\left(\phi - \frac{1}{C}\boldsymbol{V}\cdot\boldsymbol{A}\right)\right] \qquad (2.32)$$

In deriving the above equation we have used $\frac{\partial \boldsymbol{V}}{\partial x} = 0$ as velocity does not depend on x explicitly and $\frac{\partial\phi}{\partial v_x} = 0$ because ϕ is a function of x, y, z, t only and

$$\frac{d}{dt}(\boldsymbol{V}\cdot\boldsymbol{A}) = \frac{dA_x}{dt}$$

We can thus write

$$F_x = m\ddot{x} = -\frac{\partial \mathrm{U}}{\partial x} + \frac{d}{dt}\frac{\partial U}{\partial v_x} \qquad (2.32)$$

Where

$$U = q\left(\phi - \frac{1}{C}\boldsymbol{V}\cdot\boldsymbol{A}\right) \qquad (2.33)$$

Is a kind of generalized velocity dependent potential and the Lagrangian of the charged particle in the presence of electro-magnetic fields is given by

$$L = T - q\left(\phi - \frac{1}{C}\boldsymbol{V}\cdot\boldsymbol{A}\right) \qquad (2.34)$$

Example : We will illustrate this with the help of a suitable example. A particle of mass m is constrained to move under gravity on an elliptical wire in the x-y plane.

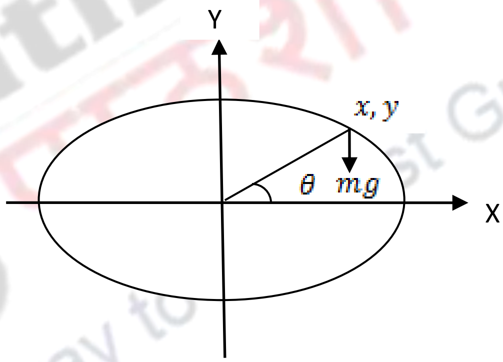
The constraint equation is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

Which can be parameterized by $x=a\cos\theta$ and $y=b\sin\theta$. Since x and y are related, let us choose the generalized coordinate to be θ. In terms of generalized coordinate the kinetic energy

$$T=\frac{1}{2}m(\dot{x}^2+\dot{y}^2)=\frac{1}{2}m\left(a^2\sin^2\theta\,\dot{\theta}^2+b^2\cos^2\theta\,\dot{\theta}^2\right)$$

The potential energy $U=mgy=mgb\sin\theta$.



The equation of motion is given by

$$\frac{d}{dt}\left(\frac{\partial T}{\partial\dot{\theta}}\right)-\frac{\partial T}{\partial\theta}=f_\theta=\frac{\partial U}{\partial\theta}$$

$$\frac{\partial T}{\partial\dot{\theta}}=\frac{1}{2}m(a^2\sin^2\theta+b^2\cos^2\theta)2\dot{\theta}$$

$$\frac{\partial T}{\partial \theta} = m\dot{\theta}^2(a^2 \sin\theta \cos\theta - b^2 \sin\theta \cos\theta)$$

$$= m \sin\theta \cos\theta\,(a^2 - b^2)\dot{\theta}^2$$

Thus the equation of motion is

$$m\ddot{\theta}(a^2 \sin^2\theta + b^2 \cos^2\theta) = -mgb\cos\theta$$

$$\ddot{\theta} = -g\frac{b\cos\theta}{a^2 \sin^2\theta + b^2\cos^2\theta}$$

For a circular wire $a = b = r$

$$\ddot{\theta} = -g\frac{\cos\theta}{r}$$

6. Summary:

- For conservative systems, generalized force can be obtained from the potential energy function expressed in terms of generalized coordinates as
$$f_j = -\frac{\partial U(\{q_k\}, t)}{\partial q_j}$$
- d'Alembert's Principle states that the rate of change of momentum is determined only by the non-constraint forces.